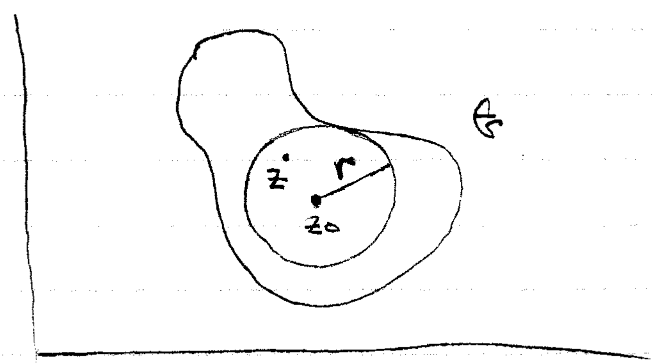
PS. 9

Taylor's Theorem

Let $f(z)$ be analytic in a region G containing the point z_0. Then the representation

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

holds (is convergent) in all disks $|z-z_0| < r$ contained in G.



Proof: Let z be an interior point of the closed disk $|\zeta - z_0| \le r$ contained in G. and use the Cauchy integral formula to write

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Writing

$$\zeta - z = (\zeta - z_0)\left[1 - \frac{z - z_0}{\underset{\shortparallel}{\zeta - z_0}}\right] \qquad \text{with } |\zeta - z_0| > |z - z_0|$$

r

$$\text{or } \left|\frac{z - z_0}{\zeta - z_0}\right| < 1$$

we have

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\cdot\frac{1}{\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}\,d\zeta$$

expand in geometric series $\quad \dfrac{1}{1-x} = 1 + x + x^2 + \cdots$

$$= \frac{1}{2\pi i}\int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\left[1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{n-1} + Q_n\right]d\zeta$$

$$\text{where } Q_n = \frac{(z - z_0)^n/(\zeta - z_0)^n}{1 - (z - z_0)/(\zeta - z_0)} = \frac{(z - z_0)^n}{(\zeta - z)(\zeta - z_0)^{n-1}}$$

~~is the remainder term~~

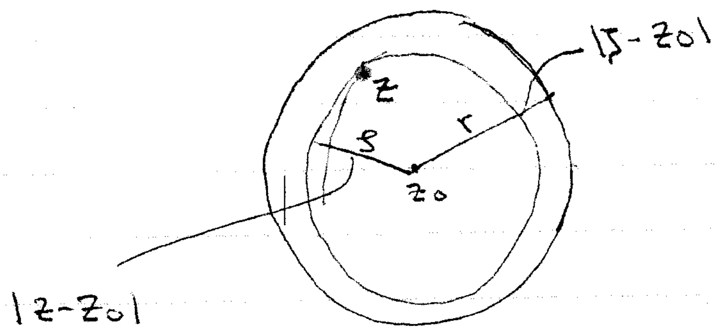
$$= \underbrace{\frac{1}{2\pi i}\int_{|\zeta - z_0| = r}\frac{f(\zeta)}{\zeta - z_0}\,d\zeta}_{f(z_0)} + \underbrace{\frac{1}{2\pi i}\int_{|\zeta - z_0| = r}\frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta}_{f'(z_0)}\,(z - z_0) + \cdots + \underbrace{\frac{1}{2\pi i}\int_{|\zeta - z_0| = r}\frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^n}}_{\frac{f^{n-1}(z_0)}{(n-1)!}}\,(z - z_0)^{n-1}$$

$$+ R_n$$

and $$R_n = \frac{(z-z_0)^n}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n}\, d\zeta$$ is the remainder term of the series.

Need to get a bound on R_n i.e show $R_n \to 0$ as $n \to \infty$

Let $|z - z_0| = \rho \quad \to \quad \rho < r = |\zeta - z_0|$



Let M be maximum of $|f(\zeta)|$ on $|\zeta - z_0| = r$

Now $|\zeta - z| = |\zeta - z_0 + z_0 - z| \geq |\zeta - z_0| - |z - z_0| = r - \rho$

$|\zeta - z| \geq r - \rho$ for all ζ on $|\zeta - z_0| = r$

$$\therefore |R_n| \leq \frac{\rho^n}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi} r M}{(r-\rho) r^n} = \frac{rM}{r-\rho}\left(\frac{\rho}{r}\right)^n$$

But $\rho/r < 1 \quad \therefore R_n \to 0$ as $n \to \infty$

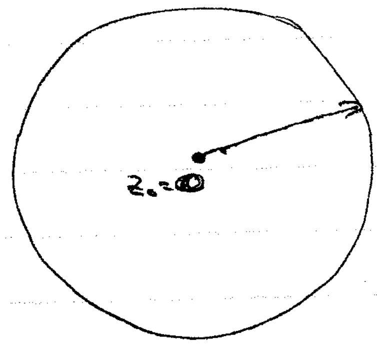
and $f(z)$ is represented by a Taylor series for all such z.

PS. 12.

i.e $$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n}$$

When $z_0 = 0$, the series representation is called a MacLaurin series i.e

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^n$$



<u>Example</u> Find MacLaurin series representation of

) $f(z) = e^z \rightarrow f^{(n)}(z) = e^z$ so $f^{(n)}(0) = 1$

and $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$ - entire fcn. valid in all $\mathbb{C}$

b) $f(z) = z^2 e^{3z}$

$$z^2 e^{3z} = z^2 \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!}$$

or let $n' = n+2$ $\quad n' = 2$ when $n = 0$

$$z^2 e^{3z} = \sum_{n'=2}^{\infty} \frac{3^{n'-2}}{(n'-2)!} z^{n'}$$ $\rightarrow$ entire fcn. valid in $\mathbb{C}$.

PS. 16

Up to now considered fcns. analytic in a domain R.

<u>Example</u> Expand the function in ~~[illegible]~~

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

in a series involving powers of z.

$$\frac{\cancel{1+2z^2}\ 1}{z^3}\left[\frac{2(1+z^2)-1}{1+z^2}\right] = \frac{1}{z^3}\left(2-\frac{1}{1+z^2}\right)$$

this is not analytic at $z=0$ so cannot expand

in a MacLaurin series about $z=0$.

But $\dfrac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 + \cdots \qquad |z|<1$

$\therefore$ when $0<|z|<1$

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots\right)$$

$$= \underbrace{\frac{1}{z^3} + \frac{1}{z}}_{\substack{\text{finite \# of terms}\\ \text{in negative powers of } z}} \underbrace{- z + z^3 - z^5 + \cdots}_{\substack{\text{looks like regular}\\ \text{MacLaurin Expansion}}}$$

$\longrightarrow$ leads to Laurent series expansion.

Laurent Series

Last series failed to be analytic at $z=0$

so Taylor's Theorem cannot be applied there.

Can often find a series representation of $f(z)$ involving

both negative and positive powers of $(z-z_0) \rightarrow$ Laurent series

Theorem: Suppose that $f(z)$ is analytic and single-valued throughout

the annular region $R_1 \leq |z-z_0| \leq R_2$ and let C

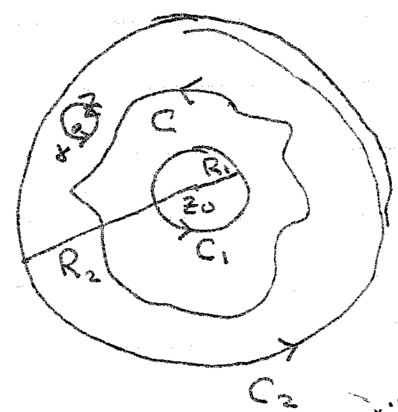
denote any positively oriented simple closed ~~curve~~ contour

around z_0 and lying in that domain.

Then at each point z in

the domain, $f(z)$ has the series

representation



$R_1 < |z-z_0| < R_2$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part}}$$

$$= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad , \quad b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

$$n = 0, 1, 2, \ldots \qquad\qquad n = 1, 2, \ldots$$

OR $$C_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

This series is called a Laurent series.

Notice that the principal part (negative powers of $(z-z_0)$) isolates the singularities or poles of $f(z)$. i.e if no poles present we recover Taylor series expansion!

$\therefore$ Laurent series e takes care of isolated singularites of $f(z)$!

Proof: Apply Cauchy's theorem for multiply connected domains

$$\int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \underbrace{\int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z}}_{= 2\pi i f(z)} = 0$$

$$\therefore\ f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$\zeta - z \to (\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]$$

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < 1$$

on C_2
usual Taylor series

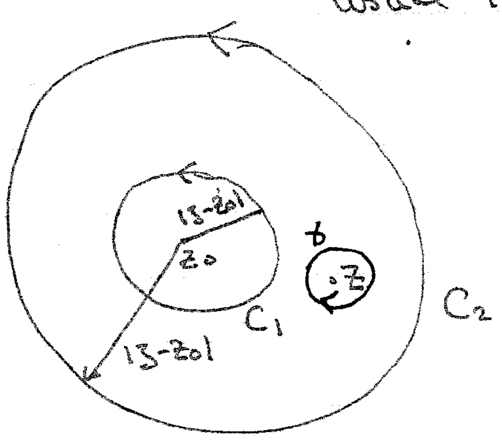
$$-\frac{1}{\zeta - z} = \frac{1}{(z - z_0) - (\zeta - z_0)}$$

$$\cancel{= \frac{1}{z - z_0}\Big[\ \Big]}$$

$$= \frac{1}{(z - z_0)\left[1 - \frac{\zeta - z_0}{z - z_0}\right]}$$

$$\left|\frac{\zeta - z_0}{z - z_0}\right| < 1$$

on C_1.



First integral is identical to Taylor series expansion.

<u>Second Integral</u>

$$-\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{z-z_0}\cdot\frac{1}{2\pi i}\int_{C_1} f(\zeta)\left[1+\left(\frac{\zeta-z_0}{z-z_0}\right)+\left(\frac{\zeta-z_0}{z-z_0}\right)^2\right.$$

$$\left. + \cdots + \left(\frac{\zeta-z_0}{z-z_0}\right)^n + \cdots\right] d\zeta$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{(n+1)}}\cdot\frac{1}{2\pi i}\int_{C_1} f(\zeta)(\zeta-z_0)^n\,d\zeta$$

Let $n' = n+1$ OR $n = n'-1$ $\quad -n = -n'+1$

$$= \sum_{n'=1}^{\infty}\frac{1}{(z-z_0)^{n'}}\cdot\underbrace{\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-n'+1}}\,d\zeta}_{b_{n'}}$$

$$= \sum_{n'=1}^{\infty}\frac{b_{n'}}{(z-z_0)^{n'}}$$

$$\therefore\ f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

because C_1 can be replaced by arbitrary curve C.
(by Cauchy's Theorem).

PS. 21

Exercise: Show that the remainder terms in each series go to zero as $n \to \infty$?

PS. 22

Laurent Series.

Examples 1

Replace z by $1/z$ in MacLaurin expansion

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots \qquad |z| < \infty$$

to get

$$e^{1/z} = 1 + 1/z + \frac{1}{2!\,z^2} + \cdots \qquad 0 < |z| < \infty$$

— contains no positive powers of z!

Note that coefficient $\boxed{c_{-1} = 1}$ (coeff of $1/z$)

But $\boxed{c_{-1} = \frac{1}{2\pi i} \int_C e^{1/z}\, dz = 1}$

Recall $\int_C (z-z_0)^n\, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$ - special significance (residue).